Slater condition (Supplementary for LP).

Consider the following convex problem:

$\mathcal{P}$. $\min f(x)$

s.t $Ax=b$, $g_i(x)\le 0$, $i=1\sim m$. $x\in\mathbb{R}^n$, $A\in\mathbb{R}^{p\times n}$

Let $D=\operatorname{dom} f\cap(\bigcap_{i=1}^{m}\operatorname{dom} g_i)$ (Recall $\operatorname{dom} f=\{x\mid f(x)<+\infty\}$)

Then if there exists $x\in \operatorname{rel}(D)$, s.t $Ax=b$, $g_i(x)<0$, $i=1\sim m$.
The strong duality holds.

Pf: W.L.O.G assume A has full row rank and $\mathcal{P}$'s optimal value is finite, denoted by p^*. and replace $\operatorname{rel}(D)$ by $\operatorname{int}(D)$.

Consider $\begin{cases}\mathbb{A}=\{(u,v,t)\mid x\in D,\ f(x)\le t,\ Ax-b=v,\ g_i(x)\le u_i,\ i=1\sim m\}\\ \mathbb{B}=\{(0,0,s)\mid s<p^*\}\end{cases}$

Then $\mathbb{A}\cap\mathbb{B}=\phi$. Since the problem is convex, $\mathbb{A}$ is convex.
Notice that $\mathbb{B}$ is convex as well.

$\Rightarrow \exists\,(\lambda,\mu,\kappa)\neq 0$, s.t $\begin{cases}\lambda^T u+\mu^T v+\kappa t\ge\alpha, & \forall(u,v,t)\in\mathbb{A}\\ \kappa s\le\alpha, & \forall(0,0,s)\in\mathbb{B}\end{cases}$, $(\exists\,\alpha)$

$\Rightarrow \kappa p^*\le\alpha$, $\lambda\ge 0$, $\kappa\ge 0$ (letting $t\to+\infty$ or $u_i\to+\infty$?)

For any $x\in D$, let $u_i=g_i(x)$, $i=1\sim m$, $v=Ax-b$, $t=f(x)$, we get:

$$\lambda^T\vec{g}(x)+\mu^T(Ax-b)+\kappa f(x)\ge\alpha\ge\kappa p^*$$

If $K \neq 0$, we get $(\frac{\lambda}{K})^T \vec{g}(x) + (\frac{\nu}{K})^T (Ax-b) + f(x) \geq p^*$ for all $x \in D$.

Hence, $\inf_{x \in D} f(x) + (\frac{\lambda}{K})^T \vec{g}(x) + (\frac{\nu}{K})^T (Ax-b) \geq p^*$ $\Rightarrow$ We get what we want.

If $K=0$, then $\lambda^T \vec{g}(x) + \nu^T (Ax-b) \geq 0$.

Since $\exists\, x_s$, s.t $\vec{g}(x) < 0$, we have: $\lambda = 0$ (Recall $\lambda \geq 0$)
Hence, $\nu^T(Ax-b) \geq 0$ for all $x \in D$. Here $\nu \neq 0$ since $(\lambda, \nu, K) \neq 0$.
$\Rightarrow \nu^T A(x-x_s) \geq 0$ for all $x \in D$.
take $x = x_s \pm \varepsilon A^T \nu$ is enough for leading contradiction for small enough ε. (A is of full row rank: AA^T is positive definite)

$\blacksquare$

(Rmk: Why we assume int(D)?
Notice that $\mathrm{aff}(D) = x_s +$ linear vector space Q. we can rewrite ous problem as

$$\min f(x_s + x - x_s)$$
$$\text{s.t } A(x-x_s) = 0,\ g_i(x_s + x - x_s) \leq 0,\ x \in D.$$

and then use Q's basis to parametrize $x - x_s$ with $\dim(Q)$ parameters, and then 0 satisfies slater condition...)

Rmk: for affine g_i, slater condition can take equality for g_i, i.e. $g_i(x) = 0$. And the strong duality still holds.

And an important observation is that: When p^* is finite (In fact, we just need the dual problem's optimal value $> -\infty$), the dual optimal value is obtained if Slater condition holds.

Now, consider

(LP) $$\min c^T x \quad \text{s.t } Ax \le b$$

$\Big\downarrow$ Dual

(DLP) $$\max -\lambda^T b \Leftrightarrow \min \lambda^T b \quad \text{s.t } \lambda \ge 0,\ A^T\lambda + c = 0$$

$\Big\downarrow$ Dual.

(DDLP) $$\min c^T x \quad \text{s.t } Ax \le b$$

If LP is feasible and bounded, we have its optimal value $p^* > -\infty$. Hence, the dual optimal value d^* is attained at some λ^* with $d^* = p^*$. Repeat this, DDLP achieves p^* at some feasible x^*